On Modeling Continuous Accelerations as Piecewise Constant Functions

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Random, non-gravitational forces acting on the spacecraft in an unpredictable manner have long been identified as a major limitation in using DSN radio data to deduce the state of the spacecraft and predict its subsequent motion. An important aspect of properly handling the non-gravitational forces is determining when their presence affects the data to an extent and in a manner that cannot be modeled accurately within the limitations of the batch filtering orbit determination procedures. This is relevant in its own right but is also important in regard to the proper configuration of the operational sequential filters. The design of these filters is such that the data is segregated into a series of batches. Between batches, stochastic elements are assumed to enter, and any or all of the parameters subject to solution can change at that time. Within any one batch, however, every parameter is assumed constant, and, within that batch, the data is treated exactly as it is treated in the classical least squares problem. In the limit as batch size reduces to a single data point, this machinery becomes identical to the point sequential filter widely discussed in the literature. To reduce the computational complexity of the operational sequential filters, however, it is desirable to keep the batch sizes as large as possible. Determining this bound in the presence of what is viewed as a continuously varying force model becomes the focus of this article.

I. Introduction

Random, non-gravitational forces acting on the spacecraft in an unpredictable manner have long been identified as a major limitation in using DSN radio data to deduce the state of the spacecraft and predict its subsequent motion (Refs. 1 and 2). Two major approaches have evolved for reducing the accuracy penalty paid for a given level of uncalibrated force involvement:

(1) Employ the net in a dual station tracking mode, taking two- and three-way doppler simultaneously. By explicitly differencing the simultaneous data, the geocentric spacecraft motion, which includes the effects of the forces and is common to both data points being received, is removed. If the stations are separated adequately, the difference of the two topocentric motions, which does not involve the random forces, is significant and can be used to determine the position of the probe. This notion of obtaining what has been termed quasi very long baseline interferometry (QVLBI) data is treated extensively by Ondrasik and Rourke (Refs. 3 and 4).

(2) Model the forces as a formal stochastic process and include this model explicitly in the filtering equations used for data processing; i.e., employ some sort of sequential filter.

These approaches are by no means mutually exclusive. In many applications, the explicit differencing operation mentioned may incur too severe a penalty in itself, because by so doing, the information inherent in the probe's geocentric motion is irretrievably lost. The existence of these forces, especially when they are present at the low levels achievable with current generation ballistic spacecraft ($<10^{-9}$ m/s^2), "clouds" the geocentric information. It does not, however, obliterate it. We are hopeful that the combination of taking simultaneous two- and three-way data and presenting these data to a sequential filter which adequately models the spacecraft forces will:

(1) Permit the inherent benefits of the QVLBI data just described to be obtained.

(2) Simultaneously enable a rather accurate reconstruction of the clouding force profile, which in turn permits the orbit determination process to capitalize on the information inherent in the long arc behavior of the probe.

An important aspect of properly handling the nongravitational forces is determining when their presence affects the data to an extent and in a manner that cannot be modeled accurately within the limitations of the batch filtering orbit determination procedures. This is relevant in its own right but is also important in regard to the proper configuration of the operational sequential filters. The design of these filters (Ref. 5) is such that the data is segregated into a series of batches. Between batches, stochastic elements are assumed to enter and any or all of the parameters subject to solution can change at that time. Within any one batch, however, every parameter is assumed constant and, within that batch, the data is treated exactly as it is treated in the classical least squares problem. In the limit as batch size reduces to a single data point, this machinery becomes identical to the point sequential filter widely discussed in the literature (see, for example, Ref. 6, Chapter 12).

To reduce the computational complexity of the operational sequential filters, however, it is desirable to keep the batch sizes as large as possible. Determining this bound in the presence of what is viewed as a continuously varying force model becomes the focus of this article. The problem is illustrated in Fig. 1, which shows a continuously varying acceleration record along with a piecewise constant representation of that function. As shown, the approximation $\hat{a}$ is allowed to change its value once every time unit. Whether this is appropriate must await a quantitative criterion.

II. Analysis

In order to create the needed criterion, Fig. 2a shows an expanded version of the first portion of the acceleration function of Fig. 1. This time we inspect the adequacy of a single, constant representation of this portion of the wave. Since the primary damage done by the stochastic forces is caused by the effects on the data (rather than direct changes of the spacecraft state) and a primary data type is doppler, it is natural to focus on the ability of the constant acceleration to produce a model velocity which tracks the actual velocity being induced. This is shown in Fig. 2b. We suggest the following criterion:

> The interval chosen over which to hold the acceleration constant should be small enough so that the rms velocity departure between the actual and resulting modeled velocity can be kept to 10^{-5} m/s or smaller.

The numerical value chosen in this criterion is the same used throughout the formulation of the Orbit Determination Program; i.e., the software model should be accurate enough to model the doppler observables to the 10^{-5} m/s accuracy level. Stated mathematically, the mean squared residual, SOS, is

$$\mathrm{SOS} = \frac{1}{T}\int_0^T [v(t) - \hat{a}t]^2\, dt \qquad (1)$$

In order to minimize Eq. (1), the following condition must be satisfied:

$$\frac{\partial \mathrm{SOS}}{\partial a} = \frac{1}{T}\int_0^T 2\,[v(t) - \hat{a}t]\,[-t]\, dt = 0 \qquad (2)$$

which implies

$$\hat{a} = \frac{3}{T^3}\int_0^T t v(t)\, dt \qquad (3)$$

The interval T must be chosen without specific knowledge of what $v(t)$ really is. The best that can be hoped for is that a reasonable *a priori* model of the random process for the non-gravitational forces can be obtained. The numerical criterion must be viewed with respect to the average SOS obtained from all possible acceleration functions drawn from the population of the process. Assume that the process is stationary, exponentially correlated with known standard deviation σ_a and correlation time τ. That is,

$$E[a(t_1)\,a(t_2)] = \sigma_a^2 e^{-|t_1-t_2|/\tau} \tag{4}$$

The statistical description of the related velocity process,

$$v(t) = \int_0^t a(\rho)\,d\rho$$

is therefore

$$E[v(t_1)\,v(t_2)] = \int_0^{t_1}\int_0^{t_2} \sigma_a^2 e^{-|\rho-\eta|/\tau}\,d\rho\,d\eta \tag{5}$$

which can be integrated to yield

$$E[v(t_1)\,v(t_2)] = \begin{cases} \sigma_a^2\tau^2\left[\left(2\dfrac{t_2}{\tau} - 1\right) + e^{-t_1/\tau} + e^{-t_2/\tau} - e^{-(t_1-t_2)/\tau}\right]; & t_2 \leq t_1 \\[2ex] \sigma_a^2\tau^2\left[\left(2\dfrac{t_1}{\tau} - 1\right) + e^{-t_1/\tau} + e^{-t_2/\tau} - e^{-(t_2-t_1)/\tau}\right]; & t_1 \leq t_2 \end{cases} \tag{6}$$

Using Eq. (3), the expected value of the (minimum) SOS can be written

$$E[\text{SOS}] = E\left[\frac{1}{T}\int_0^T v^2(t)\,dt - \frac{3}{T^4}\int_0^T\int_0^T v(\rho)\,v(\eta)\,\rho\eta\,d\rho\,d\eta\right] \tag{7}$$

After substitution of Eq. (6) into Eq. (7) and laborious algebraic detail, Eq. (7) can be shown to be

$$E[\text{SOS}] = \frac{\sigma_a^2\tau}{T}\left[\left\{4\tau^2 - \frac{5}{4}\tau T + \frac{T^2}{5} + \frac{6\tau^5}{T^3} - \frac{6\tau^3}{T}\right\} + \left\{\tau^2 - \frac{6\tau^5}{T^3} - \frac{6\tau^4}{T^2} + \frac{3\tau^3}{T}\right\}e^{-T/\tau}\right] \tag{8}$$

Letting $\alpha = \tau/T$, this equation may be rewritten as

$$E[\text{SOS}] = \sigma_a^2\tau T\left[\left\{6\alpha^5 - 6\alpha^3 + 4\alpha^2 - \frac{5}{4}\alpha + \frac{1}{5}\right\} - \alpha^2\{6\alpha^3 + 6\alpha^2 - 3\alpha - 1\}\,e^{-1/\alpha}\right] \tag{9}$$

If $T >> \tau$, then Eq. (9) reduces to

$$E[\text{SOS}] \simeq \sigma_a^2\frac{\tau T}{5}; \qquad (\alpha \text{ small}) \tag{10}$$

If $\tau > T$, then Eq. (9) becomes a very poor way to compute $E[\text{SOS}]$ as it is extremely ill-conditioned. If the exponential in Eq. (9) is expanded through eighth order and terms are grouped appropriately, it can be reduced to

$$E[\text{SOS}] \simeq \sigma_a^2\frac{T^3}{\tau}\left[\frac{1}{105} - \frac{1}{320\alpha} + \frac{1}{1512\alpha^2} + 0\left(\frac{1}{\alpha^3}\right)\right]; \qquad (\alpha \text{ large}) \tag{11}$$

which is very well-conditioned for $\tau > T$.

From formulas (10) and (11), it can be seen that

(1) $E[\text{SOS}] \to 0$ as $\tau \to \infty$

(2) $E[\text{SOS}] \to 0$ as $\tau \to 0$

This behavior is entirely reasonable due to the fact that

(1) As $\tau \to \infty$, the acceleration process becomes a bias for which the non-stochastic model can effectively track.

(2) As $\tau \to 0$, the $E[v^2(t)] \to 0$ (see Eq. 6) which implies that the velocity error due to such an acceleration process vanishes as the correlation time goes to zero.

III. Results

To observe the behavior of the expected value of the normalized sum of squares, a few details are required. First, the size of σ_a is needed. For typical missions this may be assumed to be

$$\sigma_a = 10^{-12}\,\text{km/s}^2 = 10^{-9}\,\text{m/s}^2 \tag{12}$$

Secondly, in order to scale the results in a meaningful way, it is desirable that the residual error due to the modeling be less than our stated criterion of 10^{-5} m/s. Denoting this value as e_0, let

$$\sigma_{a_s} = \frac{\sigma_a}{e_0} \tag{13}$$

In this case the scaled σ_a, σ_{a_s} will be

$$\sigma_{a_s} = 10^{-4}/s \tag{14}$$

Using this value for σ_a in formulas (9) and (11) results in E [SOS] being a dimensionless quantity, which should be less than unity if the above criterion of error is to be met. Figure 3 displays the square root of E [SOS] as a function of the correlation time τ for a family of values of the batch size T. Examination of this figure shows that if τ were equal to 5 days, as a typical example, then the batch size T would have to be 2 days or less to meet the error criterion. This also implies that a strict batch filter would begin to experience some difficulty after processing this amount of data. If, however, τ were 50 days, then T could at most be 4 days, which is somewhat surprising. This tends to indicate that the general rule-of-thumb procedure of having the batch size be some fixed fraction of the correlation time (such as 1/3) is not at all valid for a large range of values of τ.

Figure 4 displays the same information as Fig. 3; however, here the square root of E [SOS] is shown as a function of T for a family of values of τ.

As an example of how to use the charts for other circumstances, suppose the random acceleration magnitude was characteristic of solar electric missions, that is, was three orders of magnitude larger than the assumed ballistic levels of 10^{-9} m/s^2. In this case the unity level of the $\sqrt{E\,[\mathrm{SOS}]}$ would move downward 3 cycles, or reside at the bottom of the chart. Extrapolating the $\tau = 2d$ curve of Fig. 4 backwards yields that the intersection with the abscissa occurs at approximately $T = 20$ min.

The results of this analysis, displayed in these figures, should prove to be a useful aid in determining optimal batch sizes when some idea of the correlation time and error bound exists.

It should be mentioned, however, that this analysis is presented under the assumption of a single batch. The analysis for multiple batches proved far too formidable to complete. It is felt, however, that the single batch analysis should give sufficiently accurate results to serve as a useful guide.

References

1. Curkendall, D. W., et al., "The Effects of Random Accelerations on Estimation Accuracy with Applications to the Mariner 1969 Relativity Experiment," *Proceedings of the Conference on Experimental Tests of Gravitation Theories,* California Institute of Technology, Pasadena, California, November 11–13, 1970.

2. Gordon, H. J., et al., *The Mariner VI and VII Flight Paths and Their Determination from Tracking Data,* Technical Memorandum 33-469, Section II-B-6. Jet Propulsion Laboratory, Pasadena, Calif., Dec. 1, 1970.

3. Ondrasik, V. J., and Rourke, K. H., "Application of New Radio Tracking Data Types to Critical Spacecraft Navigation Problems," in *JPL Quarterly Technical Review,* Vol. 1, No. 4, pp. 116–132. Jet Propulsion Laboratory, Pasadena, Calif., Jan. 1972.

4. Rourke, K. H., and Ondrasik, V. J., "Improved Navigation Capability Utilizing Two-Station Tracking Techniques for a Low-Declination Distant Spacecraft," in *The Deep Space Network Progress Report,* Technical Report 32-1526, Vol. VII, pp. 51–60. Jet Propulsion Laboratory, Pasadena, Calif., Feb. 15, 1972.

5. Jordan, J. F., and Boggs, D., *Formulas for the Implementation of the Forward Batch-Sequential Square-Root Filter in the SATODP,* TM 391-142, Dec. 8, 1970 (JPL internal document).

6. Bryson, A. E., and Ho, Y., *Applied Optimal Control.* Blaisdell Publishing Company, Waltham, Mass., 1969.

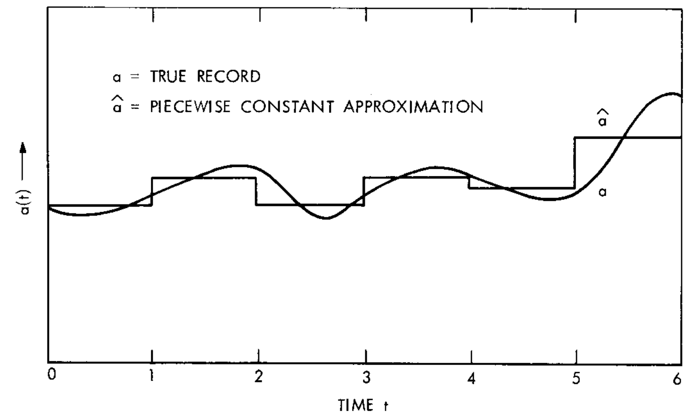

Fig. 1. Modeling a continuous record with a piecewise constant

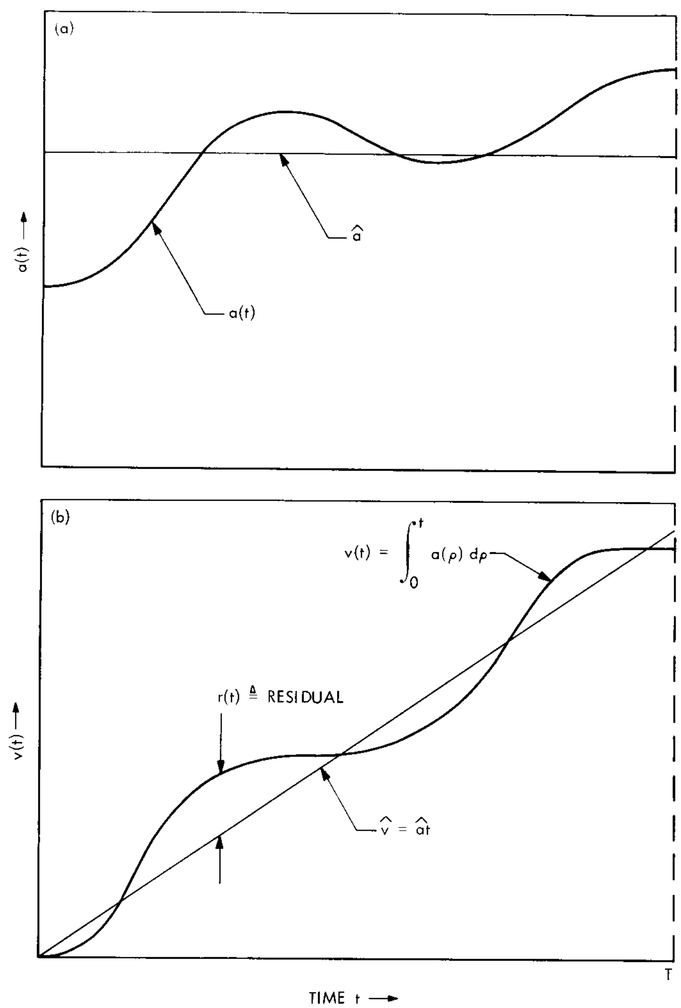

Fig. 2. First interval of Fig. 1 with velocity coordinate also displayed

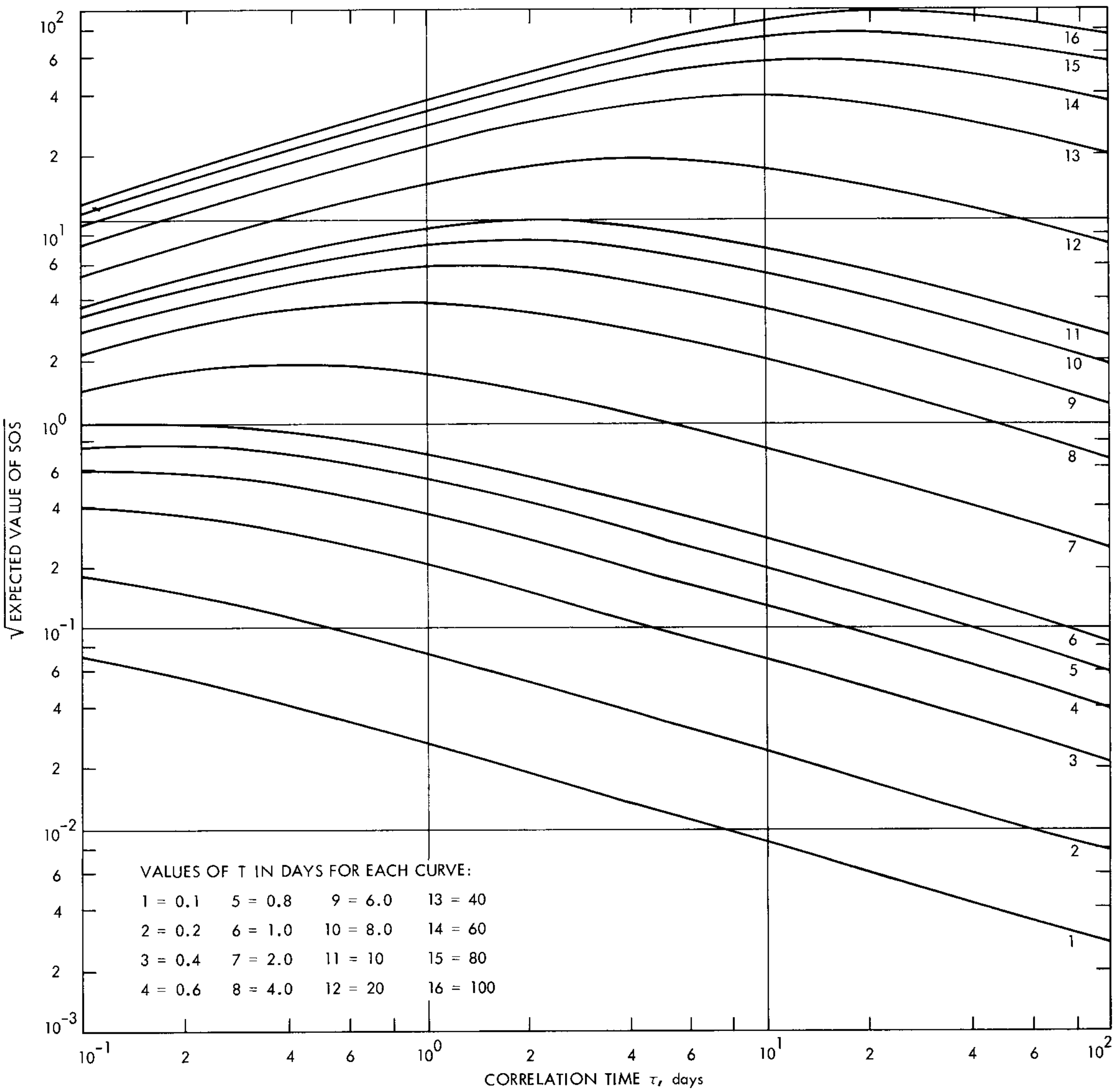

Fig. 3. The square root of *E* [SOS] as a function of correlation time for various batch sizes

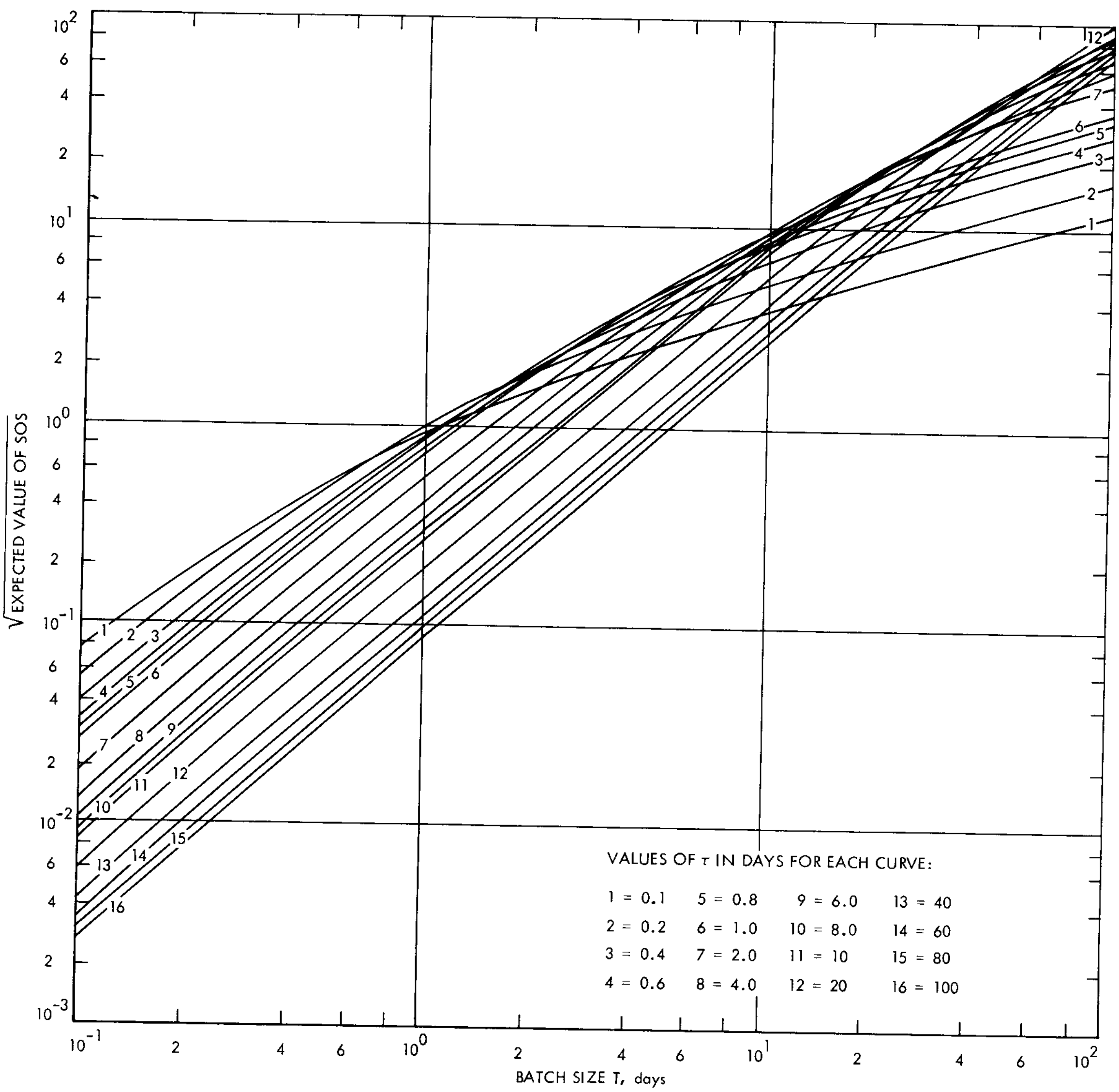

Fig. 4. The square root of E [SOS] as a function of batch size for various correlation times